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LETTER TO THE EDITOR

Exact decimation transformation with mean-field approximation

Kun-Fa Tang and Jia-Zhen Hu

Department of Applied Physics, Shanghai Jiao Tong University, Shanghai 200030, People's Republic of China

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Abstract. A new method is developed for calculating the critical temperature and exponents of spin systems, based on the exact decimation transformation and a mean-field approximation. Calculations of both exponents and critical couplings are much improved, even in the lowest approximation, when compared with other renormalisation group methods.

Real space renormalisation group methods have been widely used to study a variety of problems (see Burkhardt and van Leeuwen (1982) for reviews and detailed references). Some of the very commonly used methods for spin systems are Migdal-Kadanoff transformations (Migdal 1976, Kadanoff 1976), decimation methods (Barber 1975, Nelson and Fisher 1975, Kadanoff and Houghton 1975, Young and Stinchcombe 1976) and mean-field renormalisation group methods (Indekeu *et al* 1982). The great virtue of these methods is their simplicity, but their numerical accuracy in terms of critical temperature and exponents is generally poor. Although some improvement in the results can be achieved by using large cells, it involves lengthy calculations and slow convergence. Recently, some new approximation approaches have been proposed by several authors to improve this situation. Among these, the two-step renormalisation group (de Alcantara Bonfim *et al* 1984), which combines a decimation transformation with a mean-field renormalisation group, and cluster-decimation approximation (Goldstein *et al* 1985), which improves the Migdal-Kadanoff transformation, give much better results than those obtained by the original methods. But they are still not very good in comparison with known exact results.

In general, the renormalisation equation cannot be evaluated exactly. Starting, for example, with nearest-neighbour pair interactions only, the exact RSRG treatment will generate arbitrarily distant-neighbour and many-site interactions. Therefore, some truncating approximations, say cumulant and cluster approximations (Niemeijer and van Leeuwen 1973, 1974, Kadanoff and Houghton 1975, Barber 1975, Young and Stinchcombe 1976), have to be used in practical calculations. In this letter, we present a new idea for solving this problem. Our method yields very good results even in the lowest approximation, far exceeding in accuracy the results obtained for the same problem by other authors and which are very close to the known exact ones. It is believed that this high accuracy originates in some features of our approach where, using a mean-field approximation, an exact renormalisation transformation, which is treated by truncating the approximation in the other approaches, can be used.

In order to illustrate our approach, we consider for simplicity the case of a ferromagnetic Ising system on a square lattice, with an effective Hamiltonian

$$H=\sum_{\langle ij\rangle} K\sigma_i\sigma_j + h\sum_i \sigma_i \tag{1}$$

where $\sigma_i = \pm 1$ are the Ising spins, K is the reduced nearest coupling and h the reduced externally applied magnetic field. After the decimation transformation $(b=\sqrt{2})$ is performed, i.e. the spins denoted by dots in figure 1 are summed over, a renormalised Hamiltonian then reads (Hu 1982)

$$H'=2B\sum_{\mathrm{NN}} u_iu_j + B\sum_{\mathrm{NNN}} u_iu_j + C\sum_p u_iu_ju_ku_l + 4D\sum_i u_i + E\sum_T u_iu_ju_k + A(N^2/2) \tag{2}$$

with

$$A=\ln 2+\tfrac{1}{8}[\ln(\cosh 4K)+4\ln(\cosh 2K)]$$

$$B=\tfrac{1}{8}\ln(\cosh 4K)$$

$$C=\tfrac{1}{8}[\ln(\cosh 4K)-4\ln(\cosh 2K)]$$

$$D=\tfrac{1}{16}\{4h+\ln[\cosh(4K+h)/\cosh(h-4K)]+2\ln[\cosh(h+2K)/\cosh(h-2K)]\}$$

$$E=\tfrac{1}{16}\{\ln[\cosh(h+4K)/\cosh(h-4K)]-2\ln[\cosh(h+2K)/\cosh(h-2K)]\} \tag{3}$$

where NN denotes nearest-neighbour coupling, NNN next-nearest-neighbour coupling, p the plaquette four-spin interaction and T the three-spin interaction.

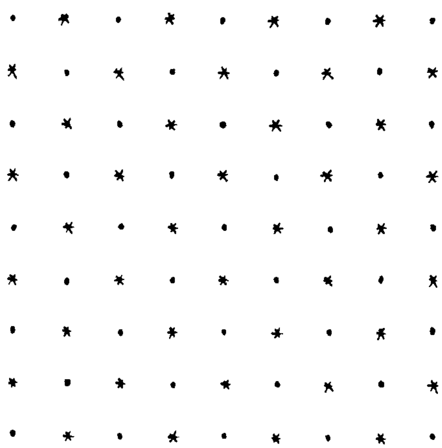

Figure 1. Decimation transformation. $b=\sqrt{2}$; $*=u_i$. Spins are alternately summed over.

It is necessary that a renormalised Hamiltonian possess an original form to find a renormalisation recursion. Although the renormalised Hamiltonian derived from the exact decimation transformation does not retain the original form, it belongs to the same universality class as the original Hamiltonian which is rewritten as follows:

$$H''=K'\sum_{\langle ij\rangle}\sigma_i\sigma_j + h'\sum_i \sigma_i. \tag{4}$$

According to the equivalence of critical behaviour of Hamiltonians H' and H'', relations between the interaction parameters of H' and H'' can be found, which, combined with equation (3), will give recursion relations for the renormalised parameters (K' and h') in terms of K and h. Here a mean-field approximation is used to perform this. Let us consider a finite cluster with L^2 spins. For this cluster with Hamiltonian H' (H''), the average magnetisation M' (M'') can be computed in the presence of symmetry-breaking boundary conditions, which, in a mean-field sense, simulate the effect of surrounding spins in infinite extensions of the finite cluster. The interactions between the spins inside the cluster with Hamiltonian H' (H'') are treated exactly, whereas the interactions between internal spins and external ones are modified by fixing the external spins to a value m' (m''). The relations between the interaction parameters of Hamiltonian H' and H'' (in the neighbourhood of $h=0$) are obtained by imposing

$$M'(B, C, D, E, m') = qM''(K', h', m'') \tag{5}$$

and

$$m' = qm'' \tag{6}$$

to leading order in h and to the first two orders in m'', for m'' approaching zero. In the lowest approximation $L=1$, equations (5) and (6) lead to

$$\begin{aligned} K' &= 3B \\ h' &= 4qE \\ q &= [1-(3C/16K^3)]^{1/2}. \end{aligned} \tag{7}$$

By using equation (3), the renormalisation group recursions become

$$\begin{aligned} K' &= \tfrac{3}{8}\ln(\cosh 4K) \\ h' &= (1+\tfrac{1}{2}\tanh 4K + \tanh 2K)hq \end{aligned} \tag{8}$$

which give the fixed point $K^* = 0.507$ and $h=0$, and the linearisation of them around the fixed point gives the critical exponents $y_T = 1.0699$ and $y_h = 2.4467$ through the relations (Niemeijer and van Leeuwen 1976)

$$\begin{aligned} \mathrm{d}K'/\mathrm{d}K &= b^{y_T} \\ \mathrm{d}h'/\mathrm{d}h &= b^{y_h}. \end{aligned} \tag{9}$$

Calculations on a sequence of clusters with $L=1$, 2 and 3 were performed to test the convergence of the method as the size of clusters increases. These results are reported in table 1. Although the clusters considered are quite small, there appears a definite and fast convergence towards exact results.

Table 1.

L	K_c	y_T	y_h
1	0.507	1.0699	2.4467
2	0.477	1.0237	1.9378
3	0.464	1.0107	1.8857
exact[a]	0.441	1.0000	1.8750

[a] Onsager (1944).

In conclusion, we have presented a new renormalisation group method, which gives very good results, even for small clusters, for universal and non-universal quantities of the two-dimensional Ising model. It should be noted that our method is very general and can be applied to any other case, say cumulant approximation, where the finite new interactions are generated in the normalisation transformation. We are applying this method to some systems of interest and hope to publish the results in the near future.

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